

RELATIVELY COMPACT SETS IN THE REDUCED C*-ALGEBRAS OF COXETER GROUPS

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ABSTRACT. We characterize relatively norm compact sets in the regular C*-algebra of finitely generated Coxeter groups using a geometrically defined positive semigroup acting on the algebra.

1. INTRODUCTION

Let (X, d) be a compact metric space, $x_0 \in X$. In $C(X)$, the continuous complex valued functions on X , consider the convex, balanced and closed set

$$\mathcal{K} = \{f : |f(x) - f(y)| \leq d(x, y), f(x_0) = 0\}.$$

The Arzela-Ascoli theorem shows that \mathcal{K} is relatively compact. On the other hand this theorem can be thought to compare any relatively compact set against this special set.

In the non-commutative context this has been made precise by Antonescu and Christensen [1] as follows:

Let A be a unital, separable C*-algebra and \mathcal{S} the set of its states endowed with the w^* -topology.

Definition 1. $\mathcal{K} \subset A$ is called a metric set if it is convex, balanced *norm compact* and *separates the states* of A

Lemma 2 ([1]). *If $\mathcal{K} \subset A$ is a metric set then*

$$d(\varphi, \psi) := \sup_{x \in \mathcal{K}} |\varphi(x) - \psi(x)|, \quad \varphi, \psi \in \mathcal{S}$$

defines a metric on \mathcal{S} , which generates the w^ -topology.*

Their *general non-commutative Arzela-Ascoli Theorem* reads as follows

Theorem 3 ([1]). *Let A be a unital C*-algebra $\mathcal{K} \subset A$ a metric set then $\mathcal{H} \subset A$ is relatively compact if and only if \mathcal{H} is bounded and for all $\epsilon > 0$ exists $N > 0$ such that*

$$\mathcal{H} \subset B_\epsilon(0) + N\mathcal{K} + \mathbb{C}Id,$$

where $B_\epsilon(0) \subset A$ is the ball of radius ϵ around 0.

Our aim here is to give an example of some such set \mathcal{K} in the reduced C*-algebra $A = C_\lambda^*(G)$ of a finitely generated Coxeter group G .

Let G, S be a Coxeter group and l the length function associated to the generating set S . (For the readers convenience in the next two sections we recall some notions and assertions related to the regular C*-algebra of Coxeter groups.)

Theorem 4.

$$\mathcal{K} = \{\lambda(f) : \|\lambda(f)\| \leq 1 \text{ and } \|\lambda(l \cdot f)\| \leq 1\}$$

is relatively compact in $C_\lambda^(G)$.*

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The proof of this theorem is given in our last section.

Since the set \mathcal{K} in $C_\lambda^*(G)$ separates the states, is convex and balanced an application of the theorem of Antonescu and Christensen characterizes relatively compact subsets of $C_\lambda^*(G)$ as follows:

Corollary 1. *A set $\mathcal{H} \subset C_\lambda^*(G)$ is relatively compact if and only if it is bounded and for all $\epsilon > 0$ there is $m \in \mathbb{N}$ such that*

$$\mathcal{H} \subset m\mathcal{K} + \mathbb{C}\lambda(\delta(e)) + B_\epsilon(0),$$

where $B_\epsilon(0) \subset C_\lambda^*(G)$ is the ball of radius ϵ and center 0.

2. COXETER GROUP

Definition 5. A pair (G, S) is a Coxeter group if, S is a finite generating subset of the group G with the following presentation:

$$\begin{aligned} s^2 &= e \quad s \in S, \\ (st)^{m(s,t)} &= e \quad s, t \in S, s \neq t, \end{aligned}$$

where $m(s, t) \in \{2, 3, 4, \dots, \infty\}$.

A specific tool for working with Coxeter groups is their *geometric representation*.

Let $V = \bigoplus_{s \in S} \mathbb{R}\alpha_s$ be an abstract real vector space with basis $\{\alpha_s : s \in S\}$. Define

a bilinear form on it: $B(\alpha_s, \alpha_t) = \begin{cases} 1 & s = t \\ -\cos \frac{\pi}{m(s,t)} & m(s,t) \neq \infty \\ -1 & m(s,t) = \infty \end{cases}$. For $s \in S$ define

a reflection by $\sigma_s \xi = \xi - 2B(\alpha_s, \xi)\alpha_s$. Then

- : $V = \mathbb{R}\alpha_s \oplus H_s$, where
- $H_s = \{\xi : B(\alpha_s, \xi) = 0\}$ is stabilized point wise by σ_s and $\sigma_s \alpha_s = -\alpha_s$.
- : $s \mapsto \sigma_s$ extends multiplicatively to a representation $\sigma : G \rightarrow \text{Gl}(V)$ of the Coxeter group.
- : σ is faithful and $\sigma(G)$ a discrete subgroup of $\text{Gl}(V)$.

We dualise the representation σ to obtain the adjoint representation

$$\sigma^*(g)f(\xi) = f(\sigma(g^{-1}\xi)), \quad f \in V^*, \xi \in V$$

For $s \in S$ let Z_s be the hyperplane $Z_s = \{f \in V^* : f(\alpha_s) = 0\}$, and A_s the halfspace $A_s = \{f \in V^* : f(\alpha_s) > 0\}$; define a family of Hyperplanes in V^* $\mathcal{H} = \bigcup_{g \in G} gZ_s$. Denote $C = \bigcap_{s \in S} A_s$ the intersection of the halfspaces, its closure $D = \overline{C} \setminus \{0\}$ is called the *fundamental chamber* usually considered as a subset of the union of its translates $U = \bigcup_{g \in G} gD$, the *Tit's cone*.

The following facts hold true:

- (i): C is a simplicial cone, its faces are the sets $Z_s \cap D$.
- (ii): U is a convex cone, D a fundamental domain for the action of G on it.
- (iii): a closed line segment $[u, c] \subset U$ meets only finitely many members of \mathcal{H} .
- (iv): Moreover, for any $c \in C$:
 $\text{card}(\{Z \in \mathcal{H} : [gc, c] \cap Z \neq \emptyset\}) = l(g),$

where $l(g) = \inf\{k : g = s_1 \dots s_k, s_i \in S\}$ denotes the usual length with respect to the generating set S . This construction due to Tits was used by Bożejko, M. and Januszkiewicz, T. and Spatzier, R. J., we recall their short proof of their theorem

Theorem 6 ([3]). *For $t > 0$*

$$\varphi_t : g \mapsto e^{-tl(g)}$$

is a positive definite function on G .

Proof.

$$\begin{aligned} l(g^{-1}h) &= \text{card}(\{Z \in \mathcal{H} : [hc, gc] \cap Z \neq \emptyset\}) \\ &= \sum_{Z \in \mathcal{H}} |\chi_h(Z) - \chi_g(Z)|^2, \end{aligned}$$

where $c \in C$ is arbitrary and χ_h is the characteristic function of $N^h = \{Z \in \mathcal{H} : [hc, c] \cap Z \neq \emptyset\}$. Hence $l(\cdot)$ is negative definite and, by a theorem of Schoenberg [9] (we only need the part already due to Schur [10]), $e^{-tl(\cdot)}$ is positive definite, see e.g. [2, Theorem 7.8]. \square

3. REGULAR REPRESENTATION

For functions $f, h : G \rightarrow \mathbb{C}$ their convolution is defined by:

$$f * h(y) = \sum_{x \in G} f(x)h(x^{-1}y).$$

For summable $f : G \rightarrow \mathbb{C}$ we denote $\lambda(f) : l^2(G) \rightarrow l^2(G)$ the associated convolution operator $\lambda(f)h = f * h$. The regular (or reduced) C^* -algebra $C_\lambda^*(G)$ is the just the operator norm closure of $\{\lambda(f) : f \in l^1(G)\}$. Denote for $g \in G$ δ_g the point mass one in $g \in G$ then $\lambda(\delta_g)$ is just left translation by g^{-1} on $l^2(G)$ and we are just dealing with the integrated version of the left regular representation. Since for $A \in C_\lambda^*(G)$ there is a unique $f = A\delta_e \in l^2$ we abuse notation to denote $A = \lambda(f)$.

The Tits cone with its deviation by the hyperplanes can be seen as a subset of a cubical building. This allows to estimate certain convolution operator norms. The first example of such an estimation was given for the free group on two generators by U. Haagerup [6] and accordingly such inequalities are called Haagerup inequality. Versions more appropriate for our purpose appear in [8, 5, 4, 11]:

Theorem 7. *A Coxeter group is a group of rapid decay: there is $C > 0$ and $k \in \mathbb{N}$ such that*

$$\|\lambda(f)\| \leq C \left(\sum_g |f(g)|^2 (1 + l(g))^{2k} \right)^{\frac{1}{2}}.$$

A consequence of this theorem is the following lemmata due to Haagerup [6, 7]. For the readers convenience we recall their proofs.

Lemma 8. *If $\varphi : G \rightarrow G$ is such that $\sup_g |\varphi(g)|(1 + l(g))^k < \infty$, then for all $\lambda(f) \in C_\lambda^*(G)$*

$$\|\lambda(\varphi \cdot f)\| \leq C \sup_g |\varphi(g)|(1 + l(g))^k \|\lambda(f)\|.$$

Here C and k are the constants in the Haagerup inequality.

Proof. From $\lambda(f)\delta_e = f$ we have $\sum_g |f(g)|^2 = \|f\|_2^2 \leq \|\lambda(f)\|^2$ and by the Haagerup inequality:

$$\begin{aligned} \|\lambda(\varphi \cdot f)\|^2 &\leq C \sum_g |\varphi(g)f(g)|^2 (1 + l(g))^{2k} \\ &\leq \sum_g |f(g)|^2 C \sup_g |\varphi(g)|^2 (1 + l(g))^{2k} \end{aligned}$$

\square

Lemma 9. *There is a sequence of finitely supported functions (ψ_m) such that for $\lambda(f) \in C_\lambda^*(G)$:*

- $\lambda(\psi_m \cdot f) \rightarrow \lambda(f)$, as $m \rightarrow \infty$
- $\|\lambda(\psi_m \cdot f)\| \leq 3\|\lambda(f)\|$

Proof. Since, by theorem 6, the functions φ_t are positive definite, they define contractive (i.e. norm non-increasing) multipliers on the regular C^* -algebra. Let

$$\varphi_{n,t} = \begin{cases} e^{-tl(g)} & \text{if } l(g) \leq n \\ 0 & \text{else} \end{cases}$$

Then

$$\begin{aligned} \|\lambda(\varphi_{n,t} \cdot f) - \lambda(f)\| &\leq \|\lambda(\varphi_{n,t} \cdot f) - \lambda(\varphi_t \cdot f)\| + \|\lambda(\varphi_t \cdot f) - \lambda(f)\| \\ &\leq C \sup_{l>n} e^{-tl}(1+l)^k \|\lambda(f)\| + \|\lambda(\varphi_t \cdot f) - \lambda(f)\| \end{aligned}$$

Since $\sup_{l>n} e^{-tl}(1+l)^k \rightarrow 0$ as $n \rightarrow \infty$ we can extract the ψ_m from the $\varphi_{n,t}$. \square

4. RELATIVELY COMPACT SETS

First we notice that the positive definite functions $\varphi_t : g \mapsto e^{-tl(g)}$ define a C_0 -semigroup of multipliers on $C_\lambda^*(G)$ given by $M_t : C_\lambda^*(G) \rightarrow C_\lambda^*(G)$, $\lambda(f) \mapsto \lambda(\varphi_t \cdot f)$.

Lemma 10. *$M : t \mapsto M_t$ is a C_0 -semigroup of contractions on $C_\lambda^*(G)$.*

Proof. Since φ_t is positive definite

$$\|M_t\| = \varphi_t(e) = 1.$$

For finitely supported f everything is elementary and now an approximation proves the assertion. \square

Lemma 11. *The generator D of the semigroup M_t is given by*

$$\begin{aligned} D(\lambda(f)) &= -\lambda(l \cdot f) \\ \text{Dom}(D) &= \{\lambda(f) : \lambda(l \cdot f) \in C_\lambda^*(G)\} \end{aligned}$$

Proof. We have

$$\lambda(\varphi_t \cdot \delta_g) = e^{-tl(g)} \lambda(\delta_g),$$

hence the assertion is clear for finitely supported $f = \sum_g f(g) \delta_g$.

Now as a generator of a C_0 -contraction semi group the operator D has a closed graph. But if

$$\lambda(f) \text{ and } \lambda(l \cdot f) \in C_\lambda^*(G),$$

then for the finitely supported ψ_m as above:

$$\lambda(\psi_m \cdot f) \rightarrow \lambda(f)$$

and

$$\lambda(\psi_m \cdot l \cdot f) \rightarrow \lambda(l \cdot f).$$

\square

Proof of Theorem 4. We shall show that for $\epsilon > 0$ there exists a finite dimensional bounded set

$$\tilde{\mathcal{K}}_\epsilon \subset C_\lambda^*(G)$$

such that for all $f \in \mathcal{K}$

$$\text{dist}(f, \tilde{\mathcal{K}}_\epsilon) \leq \epsilon.$$

(this show that \mathcal{K} is totally bounded)

We have for $f \in \mathcal{K}$:

$$\lambda(\varphi_t \cdot f) - \lambda(f) = M_t(\lambda(f)) - \lambda(f) = \int_0^t D(M_s(\lambda(f))) ds$$

Hence

$$\begin{aligned}\|\lambda(\varphi_t \cdot f) - \lambda(f)\| &\leq t \sup_{s < t} \|\lambda(le^{-sl} \cdot f)\| \\ &\leq t \|\lambda(l \cdot f)\| \leq t.\end{aligned}$$

and

$$\begin{aligned}\|\lambda(\varphi_t \cdot f) - \lambda(\varphi_{n,t} \cdot f)\| &\leq C \sup_{l > n} e^{-tl} (1+l)^k \|\lambda(f)\| \\ &\leq C \sup_{l > n} e^{-tl} (1+l)^k\end{aligned}$$

taking first t small and then n large we have an approximation to $\lambda(f)$ by certain $\lambda(\varphi_{n,t} \cdot f)$ up to ϵ uniformly in $\lambda(f) \in \mathcal{K}$. Further for this n

$$\begin{aligned}\|\lambda(\varphi_{n,t} \cdot f)\| &\leq \|\lambda(\varphi_t \cdot f) - \lambda(\varphi_{n,t} \cdot f)\| + \|\lambda(\varphi_t \cdot f)\| \\ &\leq C(\sup_{l > n} e^{-tl} (1+l)^k + 1) \|\lambda(f)\| \\ &\leq C(\sup_{l > n} e^{-tl} (1+l)^k + 1)\end{aligned}$$

So these $\lambda(\varphi_{n,t} \cdot f)$ are from a bounded set and all have their support in words of length at most n . The functions with support in this finite set give rise to a finite dimensional subspace of $C_\lambda^*(G)$. \square

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